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A spectral closure for premixed turbulent combustion in the flamelet regime

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Premixed turbulent combustion in the flamelet regime is analysed on the basis of a field equation. This equation describes the instantaneous flame contour as an isoscalar surface of the scalar field $G(\boldsymbol{x},t)$. The field equation contains the laminar burning velocity s_L as velocity scale and its extension includes the effect of flame stretch involving the Markstein length $\mathscr{L}$ as a characteristic lengthscale of the order of the flame thickness. The scalar $G(\boldsymbol{x},t)$ plays a similar role for premixed flamelet combustion as the mixture fraction $Z(\boldsymbol{x},t)$ in the theory of non-premixed flamelet combustion.

Equations for the mean $\bar{G}$ and variance $\overline{G'^2}$ are derived. Additional closure problems arise for the mean source terms in these equations. In order to understand the nature of these terms an ensemble of premixed flamelets with arbitrary initial conditions in constant-density homogeneous isotropic turbulence is considered. An equation for the two-point correlation $\overline{G'(\boldsymbol{x},t)\,G'(\boldsymbol{x}+\boldsymbol{r},t)}$ is derived. When this equation is transformed into spectral space, closure approximations based on the assumption of locality and on dimensional analysis are introduced. This leads to a linear equation for the scalar spectrum function $\Gamma(k,t)$, which can be solved analytically. The solution $\Gamma(k,t)$ is analysed by assuming a small-wavenumber cutoff at $k_0 = l_T^{-1}$, where l_T is the integral lengthscale of turbulence. There exists a $k^{-\frac{5}{3}}$-spectrum between l_T and L_G, where L_G is the Gibson scale. At this scale turbulent fluctuations of the scalar field $G(\boldsymbol{x},t)$ are kinematically restored by the smoothing effect of laminar flame propagation. A quantity called kinematic restoration ω is introduced, which plays a role similar to the scalar dissipation χ for diffusive scalars.

By calculating the appropriate moments of $\Gamma(k,t)$, an algebraic relation between ω, $\overline{G'(\boldsymbol{x},t)^2}$, the integral lengthscale l_T and the viscous dissipation ϵ is derived. Furthermore, the scalar dissipation $\chi_{\mathscr{L}}$, based on the Markstein diffusivity $\mathscr{D}_{\mathscr{L}} = s_L\mathscr{L}$, and the scalar-strain co-variance $\Sigma_{\mathscr{L}}$ are related to ω. Dimensional analysis, again, leads to a closure of the main source term in the equation for the mean scalar $\bar{G}$. For the case of plane normal and oblique turbulent flames the turbulent burning velocity s_T and the flame shape is calculated. In the absence of flame stretch the linear relation $s_T \sim u'$ is recovered. The flame brush thickness is of the order of the integral lengthscale. In the case of a V-shaped flame its increase with downstream position is calculated.

1. Introduction

Ever since Damköhler (1940) defined the turbulent burning velocity s_T in analogy with the laminar burning velocity s_L as the velocity of a turbulent premixed flame relative to the mean flow, the question has remained of how s_T depends on the turbulence intensity u', the laminar burning velocity s_L and on turbulent and laminar lengthscales. If the flame structure is assumed to be a surface of discontinuity with zero thickness, then there exists no reference length and dimensional arguments suggest that the turbulent lengthscale should not enter into the problem. Damköhler has considered this limit and has equated the mass flux $\dot{m}$ of unburned gas through the instantaneous wrinkled flame surface area F_{T} to the mass flux through the cross-sectional area F

$$\dot{m} = \rho_{\mathrm{u}} s_{\mathrm{L}} F_{\mathrm{T}} = \rho_{\mathrm{u}} s_{\mathrm{T}} F, \tag{1.1}$$

where ρ_{u} is the density of the unburnt gas. He argued that the flow through the instantaneous flame surface F_{T} proceeds with the laminar burning velocity s_{L}, while that through the cross-sectional area has the turbulent burning velocity s_{T}. Using the geometrical analogy with a Bunsen flame, Damköhler assumed that the area increase of the wrinkled flame surface area relative to the cross-sectional area is proportional to the increase u' of flow velocity relative to the laminar burning velocity

$$\frac{F_{\mathrm{T}}}{F} = \frac{s_{\mathrm{L}} + u'}{s_{\mathrm{L}}}. \tag{1.2}$$

Finally, the velocity increase was assumed proportional to the turbulence intensity u'. Combining (1.1) and (1.2) leads to

$$\frac{s_{\mathrm{T}}}{s_{\mathrm{L}}} = 1 + \frac{u'}{s_{\mathrm{L}}}. \tag{1.3}$$

In the limit $u' \gg s_{\mathrm{L}}$ one obtains

$$s_{\mathrm{T}} \sim u', \tag{1.4}$$

which is Damköhler's result. It states that the turbulent burning velocity should only depend on the turbulence intensity and should be independent of laminar flame properties like s_{L} or the laminar flame thickness l_{F}. Since these depend on laminar transport properties and chemistry, neither should influence the process of turbulent flame propagation. This is inconsistent with experimental data, which show a dependence of $s_{\mathrm{T}}/s_{\mathrm{L}}$ on fuel composition, turbulent and laminar lengthscales and a nonlinear dependence on the ratio u'/s_{L}. Recent compilations of literature data will be referenced below.

Theoretical approaches to resolve the problem of turbulent flame propagation are very often based on closure assumptions for the chemical source term in the balance equations of the mean scalar quantities such as temperature and concentrations. The most prominent of these is the eddy-breakup model (Mason & Spalding 1973). Modifications of this model were proposed by Magnussen & Hjertager (1977), Borghi & Dutoya (1979), Bray (1979), Bray, Libby & Moss (1984), and recently by Bray, Champion & Libby (1988), Gouldin, Bray & Chen (1989), Cant & Bray (1989), Borghi (1990), and Catlin & Lindstedt (1991). All these models recover Damköhler's result to leading order, and some of them introduce empirical corrections accounting for flame stretch.

Differently from non-premixed combustion, a consistent flamelet formulation is still missing for premixed combustion. In non-premixed combustion the problem could be formulated (Peters 1984) by separately considering an equation for the mixture fraction Z, which fixes the instantaneous location of the flamelets at $Z(\boldsymbol{x}, t) = Z_{\mathrm{st}}$, and flamelet equations in terms of Z as independent variable. Only the latter equations contain chemical source terms and molecular diffusion. A turbulent timescale is imposed on the flamelets by the outer field and is parameterized by the local value of the scalar dissipation χ_{st}. A similar formulation for premixed

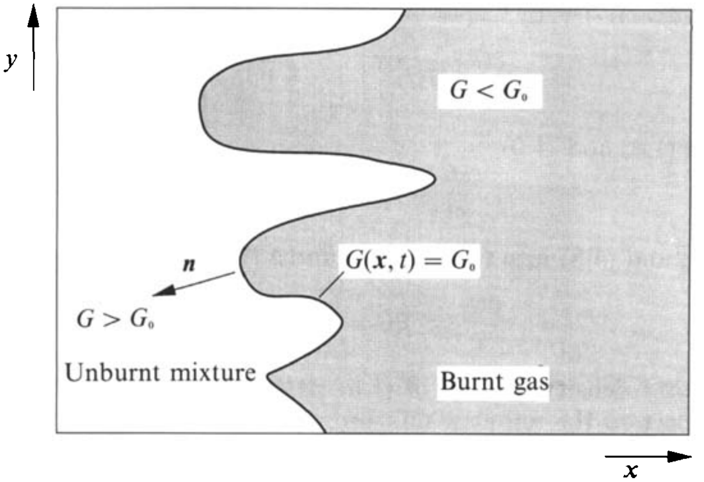

FIGURE 1. A schematic representation of the flame front as an isoscalar surface of the scalar $G(\boldsymbol{x}, t)$.

combustion should at first contain an analogous scalar equation, which fixes the location of the flamelets. Such an equation for the scalar $G(\boldsymbol{x}, t)$ whose level surfaces $G(\boldsymbol{x}, t) = G_0$ represent the flame surface, is the equation derived by Williams (1985)

$$\rho\left(\frac{\partial G}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}G\right) = (\rho s_{\mathrm{L}})\,|\boldsymbol{\nabla}G|, \tag{1.5}$$

where (ρs_{L}) is the constant-mass burning velocity through a laminar, plane steady flame. The properties of this equation have been investigated in a number of papers. Kerstein, Ashurst & Williams (1988) and Ashurst (1990) have solved it numerically in two- and three-dimensional turbulent flow fields, while Yakhot (1988*a*) and Sivashinsky (1989) have applied renormalization group analysis to it. Yakhot (1988*b*) has analysed the scaling properties of (1.5) and has argued that it would lead to a linear dependence of the turbulent on the laminar burning velocity. Recently Vassilicos (1990) has investigated the behaviour of solutions for small times.

For the physical interpretations below it should be interesting to show that (1.5) may be derived from the local kinematic relation between the propagation velocity $\boldsymbol{v}_{\mathrm{p}}$ of a flame front, the flow velocity $\boldsymbol{v}$ and the laminar burning velocity s_{L}

$$\boldsymbol{v}_{\mathrm{p}}\cdot\boldsymbol{n} = \boldsymbol{v}\cdot\boldsymbol{n}+s_{\mathrm{L}}. \tag{1.6}$$

The condition for the flame front

$$G(\boldsymbol{x}, t) = G_0 \tag{1.7}$$

divides the flow field into two regions where $G > G_0$ is the region of burnt gas and $G < G_0$ that of the unburnt mixture (cf. figure 1). The normal vector on the surface towards the unburnt is then given by

$$\boldsymbol{n} = -\frac{\boldsymbol{\nabla}G}{|\boldsymbol{\nabla}G|}. \tag{1.8}$$

The local propagation velocity $\boldsymbol{v}_{\mathrm{p}}$ of the front is defined by

$$\boldsymbol{v}_{\mathrm{p}} = \left.\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|_{G=G_0}. \tag{1.9}$$

If one differentiates (1.7) with respect to t,

$$\frac{\partial G}{\partial t}+\nabla G\frac{\partial \boldsymbol{x}}{\partial t}\bigg|_{G=G_0}=0, \tag{1.10}$$

one obtains with (1.8) and (1.9)

$$\frac{\partial G}{\partial t}=\boldsymbol{v}_{\mathrm{p}}\cdot\boldsymbol{n}\,|\nabla G|. \tag{1.11}$$

Introducing (1.6) and (1.8) into (1.11) one obtains the field equation

$$\frac{\partial G}{\partial t}+\boldsymbol{v}\cdot\nabla G=s_{\mathrm{L}}|\nabla G|. \tag{1.12}$$

This is the constant-density version of (1.5). If the burning velocity s_{L} in (1.6) is defined with respect to the unburnt mixture, then the flow velocity $\boldsymbol{v}$ in (1.6) and (1.12) is also defined as the conditioned velocity field in the unburnt mixture ahead of the flame. Comparison of (1.12) with (1.5) shows that the density ρ in (1.5) should then be equal to ρ_{u}. On the other hand, if s_{L} is the burning velocity with respect to the burnt gas, then $\boldsymbol{v}$ and ρ in (1.5) should be the velocity and the density in the burnt gas.

Since (1.5) represents a balance of velocities only, it contains no lengthscale. The characteristic lengthscale for flame response is the Markstein length $\mathscr{L}$ introduced by Pelce & Clavin (1982), Clavin (1985). It is proportional to the flame thickness. For the case of a one-step large-activation-energy reaction and a constant thermal conductivity λ, dynamic viscosity μ and heat capacity c_p, the ratio of $\mathscr{L}$ to the flame thickness l_{F} is

$$\frac{\mathscr{L}}{l_{\mathrm{F}}}=\frac{1}{\gamma}\ln\frac{1}{1-\gamma}+\frac{\beta(Le-1)}{2}\frac{(1-\gamma)}{\gamma}\int_0^{\gamma/(1-\gamma)}\frac{\ln(1+x)}{x}\mathrm{d}x. \tag{1.13}$$

This expression was first derived by Clavin & Williams (1982). Here $\gamma=(T_{\mathrm{b}}-T_{\mathrm{u}})/T_{\mathrm{b}}$, where T_{b} and T_{u} are the temperatures in the burnt and the unburnt gas, respectively; $\beta=E(T_{\mathrm{b}}-T_{\mathrm{u}})/RT_{\mathrm{b}}^2$ is the Zeldovich number, where E is the activation energy and R the universal gas constant; and $Le=\lambda/\rho c_p D$ is the Lewis number of the reactant, assumed constant, where D is the molecular diffusivity. The flame thickness is $l_{\mathrm{F}}=\lambda/(\rho c_p s_{\mathrm{L}})$. Equation (1.13) was derived with respect to the unburnt mixture. Clavin (1985) shows that with respect to the burnt gas one obtains a similar but different expression where the factor $(1-\gamma)$ in the nominator of the second term is missing. This increases the influence of Lewis number effects since the term $(1-\gamma)$ is equal to $T_{\mathrm{u}}/T_{\mathrm{b}}$, which is typically between 0.15 and 0.2 in technical flames. It should be noted that the Markstein length changes if more than one reaction is considered. Experimental values for the ratio in (1.13) range typically from $\mathscr{L}/l_{\mathrm{F}}=2$ to $\mathscr{L}/l_{\mathrm{F}}=6$ (Searby & Quinard 1990).

The Markstein length is introduced into (1.12) by using the burning velocity of a flame submitted to stretch. Flame stretch consists of two contributions: one due to flame curvature and another due to flow divergence. The burning velocity s_{L} is modified by these two effects as

$$s_{\mathrm{L}}=s_{\mathrm{L}}^0-s_{\mathrm{L}}^0\mathscr{L}\kappa+\mathscr{L}\boldsymbol{n}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{n}. \tag{1.14}$$

Here s_{L}^0 is the burning velocity for an unstretched flame. The flame curvature κ is defined

$$\kappa=\nabla\cdot\boldsymbol{n}=-\nabla\cdot\left(\frac{\nabla G}{|\nabla G|}\right), \tag{1.15a}$$

which may be transformed as

$$\kappa = -\frac{\nabla^2 G}{|\nabla G|} + \frac{\nabla|\nabla G|\cdot\nabla G}{|\nabla G|^2} = -\frac{\nabla^2 G - \boldsymbol{n}\cdot\nabla|\nabla G|}{|\nabla G|}. \tag{1.15b}$$

The second term in the numerator may be combined with the last term in (1.14) which, when multiplied with $|\nabla G|$, leads to

$$s_L^0\,\boldsymbol{n}\cdot\nabla|\nabla G| - \boldsymbol{n}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{n}|\nabla G| = -\boldsymbol{n}\cdot\left(\nabla\left(\frac{\mathrm{D}G}{\mathrm{D}t}\right) - \nabla\boldsymbol{v}\cdot\nabla G\right)$$

$$= -\boldsymbol{n}\cdot\left[\frac{\mathrm{D}(\nabla G)}{\mathrm{D}t}\right]_{\mathscr{L}\to 0} = \left[\frac{\mathrm{D}|\nabla G|}{\mathrm{D}t}\right]_{\mathscr{L}\to 0}. \tag{1.16}$$

Here (1.12) has been used with $s_L = s_L^0$ and the substantial derivative is defined as

$$\frac{\mathrm{D}G}{\mathrm{D}t} = \frac{\partial G}{\partial t} + \boldsymbol{v}\cdot\nabla G. \tag{1.17}$$

Therefore, introducing the burning velocity correction due to stretch into (1.12) leads to the following equation for the scalar $G(\boldsymbol{x}, t)$:

$$\frac{\mathrm{D}G}{\mathrm{D}t} \equiv s_L^0|\nabla G| + \mathscr{D}_{\mathscr{L}}\nabla^2 G - \mathscr{L}\left[\frac{\mathrm{D}|\nabla G|}{\mathrm{D}t}\right]_{\mathscr{L}\to 0}. \tag{1.18}$$

Here, $\mathscr{D}_{\mathscr{L}} = s_L^0\,\mathscr{L}$ is the corresponding Markstein diffusivity. Equation (1.18) represents a first-order expansion for $\mathscr{L}\to 0$ of (1.12) and has been derived by a two-scale asymptotic analysis by Peters & Keller (1992). It could also have been anticipated from equation (6.1) in Matalon & Matkowsky (1982) or equation (23) in Clavin & Joulin (1983), written in isotropic and dimensional form. It is a so-called Hamilton–Jacobi equation with a parabolic second order differential operator coming from the curvature term.

Another lengthscale that will be important for the subsequent analysis is the Gibson length (Peters 1986)

$$L_G = \frac{(s_L^0)^3}{\epsilon}, \tag{1.19}$$

where ϵ is the turbulent dissipation. It represents a lower cutoff scale for the interaction of turbulent eddies with a flame front. While eddies much larger than L_G convect the flame surface as if it was a passive surface, much smaller eddies cannot kinematically counterbalance the flame propagation since their velocity difference is much smaller than s_L. The physical processes that occur at the Gibson length will play a fundamental role in the analysis that follows.

2. Equations for the mean and the variance of the scalar G

We start from (1.18), which can be written, without loss of generality, for a Cartesian coordinate system, where Greek subscripts appearing twice indicate a summation from 1 to 3. The differential operators may be written as

$$\frac{\mathrm{D}G}{\mathrm{D}t} \equiv \frac{\partial G}{\partial t} + v_\alpha\frac{\partial G}{\partial x_\alpha}, \quad \nabla^2 G \equiv \frac{\partial^2 G}{\partial x_\alpha^2} \tag{2.1}$$

and in the following, the modulus of G will be denoted by

$$\sigma \equiv |\nabla G| \equiv \left[\left(\frac{\partial G}{\partial x_\beta}\right)^2\right]^{\frac{1}{2}}. \tag{2.2}$$

The last term in (1.18) may be expressed in terms of spatial gradients only. By differentiating the leading-order equation (1.12) with s_{L} replaced by s_{L}^0 with respect to x_β, defining $\sigma_\beta = \partial G/\partial x_\beta$ and multiplying the equation for σ_β by $2\sigma_\beta$ one obtains

$$\left(\frac{\partial \sigma_\beta^2}{\partial t} + v_\alpha \frac{\partial \sigma_\beta^2}{\partial x_\alpha} + 2\frac{\partial v_\alpha}{\partial x_\beta}\sigma_\beta \sigma_\alpha\right) = 2s_{\mathrm{L}}^0 \frac{\partial \sigma}{\partial x_\beta}\frac{G}{\partial x_\beta}. \tag{2.3}$$

With $\sigma^2 = \sigma_\beta^2$ one obtains

$$\frac{\mathrm{D}\sigma}{\mathrm{D}t} = s_{\mathrm{L}}^0 \frac{\partial \ln \sigma}{\partial x_\beta}\frac{\partial G}{\partial x_\beta} - \frac{\partial v_\alpha}{\partial x_\beta}\frac{\sigma_\beta \sigma_\alpha}{\sigma}. \tag{2.4}$$

Therefore (1.18) may be written as

$$\left(\frac{\partial G}{\partial t} + v_\alpha \frac{\partial G}{\partial x_\alpha}\right) = s_{\mathrm{L}}^0 \sigma + \mathscr{D}_{\mathscr{L}} \frac{\partial^2 G}{\partial x_\alpha^2} - \mathscr{D}_{\mathscr{L}} \frac{\partial \ln \sigma}{\partial x_\beta}\frac{\partial G}{\partial x_\beta} + \mathscr{L}\frac{\partial v_\alpha}{\partial x_\beta}\frac{\sigma_\beta \sigma_\alpha}{\sigma}. \tag{2.5}$$

The isoscalar surface $G(\boldsymbol{x}, t) = G_0$, where G_0 is an arbitrary constant, describes the instantaneous flame contour. The scalar difference $G - G_0$ may be interpreted as the distance from the flame surface. For dimensional analysis, however, the properties of the scalar equation must be treated as those of any scalar, the temperature for instance, and the dimension of G will be denoted by [g].

Equation (2.5) may be treated as any scalar equation in a turbulent flow field. In particular, G and v_α may be split into a mean and a fluctuation

$$G = \bar{G} + G', \quad v_\alpha = \overline{v_\alpha} + v_\alpha'. \tag{2.6}$$

The equation for the mean is

$$\left(\frac{\partial \bar{G}}{\partial t} + \overline{v_\alpha}\frac{\partial \bar{G}}{\partial x_\alpha}\right) + \frac{\partial}{\partial x_\alpha}(\overline{v_\alpha' G'}) = s_{\mathrm{L}}^0 \bar{\sigma} + \mathscr{D}_{\mathscr{L}} \frac{\partial^2 \bar{G}}{\partial x_\alpha^2} - \mathscr{D}_{\mathscr{L}} \overline{\frac{\partial \ln \sigma}{\partial x_\beta}\frac{\partial G}{\partial x_\beta}} + \mathscr{L}\overline{\frac{\partial v_\alpha}{\partial x_\beta}\frac{\sigma_\beta \sigma_\alpha}{\sigma}}. \tag{2.7}$$

In the following we will assume that s_{L}^0 and $\mathscr{L}$ are defined with respect to the unburnt mixture. Then v_α is the conditioned velocity field ahead of the flame. The equation for the variance is obtained by multiplying the equation for the fluctuation G', which is obtained by subtracting (2.7) from (2.5), by $2G'$ and averaging:

$$\left(\frac{\partial \overline{G'^2}}{\partial t} + \overline{v_\alpha}\frac{\partial \overline{G'^2}}{\partial x_\alpha}\right) = -\frac{\partial}{\partial x_\alpha}(\overline{v_\alpha' G'^2}) + -2\overline{v_\alpha' G'}\frac{\partial \bar{G}}{\partial x_\alpha} - \omega - \chi_{\mathscr{L}} - \Sigma_{\mathscr{L}}. \tag{2.8}$$

Here the first term on the right-hand side denotes the turbulent transport of the scalar variance and the second its production by turbulent fluctuations. The last three terms are specific for the present problem and are defined as follows.

Kinematic restoration:

$$\bar{\omega} = -2s_{\mathrm{L}}^0 \overline{G'\sigma'}. \tag{2.9}$$

This term represents the co-variance of the scalar fluctuation with the source term in (1.18). This co-variance is expected to be negative. This may be illustrated by arguments similar to those of Damköhler (1940). Let us consider a two-dimensional steady oblique flame with a constant velocity u in the x direction, where $u > s_L^0$. For this configuration the solution of (1.18) in the limit $\mathscr{L} = 0$ is

$$G = \frac{(u^2 - s_L^{0^2})^{\frac{1}{2}}}{s_L^0}|y| + x - x_0, \tag{2.10}$$

with symmetry with respect to the x-axis and the flame tip lying at $x = x_0$, $y = 0$. Superimposing on this configuration a velocity fluctuation, one realizes that a small velocity increase u' would decrease the flame angle with respect to the x-axis thereby introducing a positive fluctuation σ'. At the same time the flame front is convected downstream which induces a negative fluctuation G'. The opposite is true for a small velocity decrease. Therefore the product $G'\sigma'$ is negative in both cases and the correlation $\overline{G'\sigma'}$ is also likely to be negative. The kinematic restoration accounts for the smoothing of the scalar field and thereby of the flame surface by laminar flame propagation. Scalar fluctuations produced by turbulence are restored by this kinematic effect, which is most effective at the Gibson lengthscale L_G. The physical interpretation of this effect has been given by Peters (1986). The kinematic restoration will play a central role in the analysis that follows.

Scalar dissipation:

$$\chi_{\mathscr{L}} = -2\mathscr{D}_{\mathscr{L}}\overline{G'\left(\frac{\partial^2 G}{\partial x_\alpha^2} - \frac{\partial \ln \sigma}{\partial x_\beta}\frac{\partial G}{\partial x_\beta}\right)'}. \tag{2.11}$$

This term incorporates the co-variance between G' and all diffusive terms in (1.18). These terms account for the curvature effect, which smooths the cusps formed when two parts of the flame front intersect. For constant density it can easily be decomposed into a negative diffusion term, a positive scalar dissipation term and additional nonlinear contributions. It is assumed that the scalar dissipation term is dominant and the entire term is called scalar dissipation for convenience. It will be shown to be most effective at a Corrsin lengthscale

$$L_C = (\mathscr{D}_{\mathscr{L}}^3/\epsilon)^{\frac{1}{4}} = L_G^{\frac{1}{4}}\mathscr{L}^{\frac{3}{4}} \tag{2.12}$$

based on the Markstein diffusivity $\mathscr{D}_{\mathscr{L}}$.

Scalar-strain co-variance:

$$\Sigma_{\mathscr{L}} = -2\mathscr{L}\overline{G'\left(\frac{\partial v_\alpha}{\partial x_\beta}\frac{\sigma_\beta \sigma_\alpha}{\sigma}\right)'}. \tag{2.13}$$

This term accounts for the co-variance between G' and the stretch term due to flow divergence in (1.18). As the straining motion of the turbulent flow field acts on the flame surface preferentially by stretching it (rather than by compressing it), this term smooths the flame front further and thereby reduces the remaining scalar fluctuations. It therefore is expected to be positive and the co-variance negative. It is most effective at the Markstein length $\mathscr{L}$ as will be shown below.

3. Two-point scalar correlations and their spectrum function

Closure of the problem is achieved if the unknown source terms in (2.7) and (2.8) can be expressed as functions of $\bar{G}$, $\overline{G'^2}$ and parameters representing the turbulent flow field. In order to obtain some guidance for the modelling of these terms, we consider a scalar field $G(\boldsymbol{x}, t)$ in constant-density homogeneous isotropic turbulence.

This corresponds to an ensemble of premixed flamelets with arbitrary initial conditions travelling from all directions into a given spatial domain. Since translational invariance of a premixed flame implies an arbitrary starting time, the choice of G_0 then determines the average time interval of propagation for flamelets to arrive in the spatial domain considered. We shall consider the spatial correlation between scalar fluctuations $G'(\boldsymbol{x},t)$ at point $\boldsymbol{x}$ and $G'(\boldsymbol{x}+\boldsymbol{r},t)$ at point $\boldsymbol{x}+\boldsymbol{r}$. Because of isotropy it is obvious that the correlation must simply be a scalar function of $r=|\boldsymbol{r}|$, the distance between the two points. We define

$$g^2(r,t)=\overline{G'(\boldsymbol{x},t)\,G'(\boldsymbol{x}+\boldsymbol{r},t)}. \tag{3.1}$$

An equation for g^2 can be derived by using standard techniques in homogeneous turbulence (Batchelor 1953)

$$\frac{\partial g^2}{\partial t}+2\frac{\partial \overline{v'_\alpha(\boldsymbol{x}+\boldsymbol{r},t)\,G'(\boldsymbol{x},t)\,G'(\boldsymbol{x}+\boldsymbol{r},t)}}{\partial r_\alpha}+2s_{\mathrm{L}}^0 S_1+2\mathscr{D}_{\mathscr{L}}S_2+2\mathscr{L}S_3=0, \tag{3.2}$$

where

$$\left.\begin{aligned}
S_1(r,t)&=-\overline{G'(\boldsymbol{x}+\boldsymbol{r},t)\,\sigma'(\boldsymbol{x},t)}\\
S_2(r,t)&=-\overline{G'(\boldsymbol{x}+\boldsymbol{r},t)\left[\frac{\partial^2 G(\boldsymbol{x},t)}{\partial x_\alpha^2}-\frac{\partial \ln\sigma(\boldsymbol{x},t)}{\partial x_\beta}\frac{\partial G(\boldsymbol{x},t)}{\partial x_\beta}\right]'}\\
S_3(r,t)&=-\overline{G'(\boldsymbol{x}+\boldsymbol{r},t)\left[\frac{\partial v_\alpha(\boldsymbol{x},t)}{\partial x_\beta}\frac{\sigma_\beta(\boldsymbol{x},t)\,\sigma_\alpha(\boldsymbol{x},t)}{\sigma(\boldsymbol{x},t)}\right]'}.
\end{aligned}\right\} \tag{3.3}$$

In (3.2), because of homogeneity, all spatial gradients $\partial/\partial x_\alpha$ of averaged quantities have been neglected and only the gradients in correlation space $\partial/\partial r_\alpha$ have been retained. Equation (3.2) is the analogue of Corrsin's (1951) equation for a diffusive scalar. In particular, symmetry conditions for the triple correlations

$$\overline{v'_\alpha\,G'(\boldsymbol{x},t)\,G'(\boldsymbol{x}+\boldsymbol{r},t)}=-\overline{v'_\alpha(\boldsymbol{x}+\boldsymbol{r},t)\,G'(\boldsymbol{x},t)\,G'(\boldsymbol{x}+\boldsymbol{r},t)} \tag{3.4}$$

and for any correlation of scalars such as

$$\overline{G'(\boldsymbol{x},t)\,\sigma(\boldsymbol{x}+\boldsymbol{r})}=\overline{G'(\boldsymbol{x}+\boldsymbol{r},t)\,\sigma(\boldsymbol{x},t)} \tag{3.5}$$

have been used. Since no modelling hypothesis seems evident at this stage, (3.2) will be transformed into Fourier space (Batchelor 1953). The Fourier transform is defined

$$\widehat{g^2}(\boldsymbol{k},t)=\frac{1}{(2\pi)^3}\int_{V(r)} g^2(\boldsymbol{r},t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{r}, \tag{3.6}$$

where integration is performed over a volume in correlation space. The scalar spectrum function $\Gamma(k,t)$ is related to $\widehat{g^2}(\boldsymbol{k},t)$ by

$$\Gamma(k,t)=k^2\oint \widehat{g^2}(\boldsymbol{k},t)\,\mathrm{d}\Omega=4\pi k^2\widehat{g^2}(k,t), \tag{3.7}$$

where $\mathrm{d}\Omega$ is the solid angle and $k=|\boldsymbol{k}|$ is the absolute value of the wavenumber. Fourier transformation of the triple-correlation term and the three source terms in (3.2) leads to the following equation for the scalar spectrum function:

$$\frac{\partial \Gamma(k,t)}{\partial t}-T(k,t)+4\pi k^2[2s_{\mathrm{L}}^0\widehat{S_1}(k,t)+2\mathscr{D}_{\mathscr{L}}\widehat{S_2}(k,t)+2\mathscr{L}\widehat{S_3}(k,t)]=0. \tag{3.8}$$

In this equation $T(k,t)$ represents the spectral transfer of scalar fluctuations from all other wavenumbers by turbulence in a similar way as for a diffusive scalar (cf. Monin & Yaglom 1975, p. 138).

In order to proceed further in the present analysis, we extend the hypothesis about the existence of an equilibrium range to the scalar field under consideration. We furthermore expect that there exists an inertial subrange between the wavenumbers k_0 and k_G, where Γ has an universal form and is quasi-stationary. Here k_0 is the inverse of the integral lengthscale l_T, and k_G the inverse of the Gibson scale, i.e.

$$k_0 \ll k \ll k_G, \quad l_T \gg k^{-1} \gg L_G. \tag{3.9}$$

To calculate the scalar spectrum, we will need to adopt a closure hypothesis for the scalar transfer function $T(k,t)$. For simplicity we use the gradient transport assumption introduced by Pao (1965, 1968) for a diffusive scalar (cf. Monin & Yaglom 1975, p. 406), which we write as $T(k,t) = -\partial W(k,t)/\partial k$, where

$$W(k,t) = C_S^{-1}\,\epsilon^{\frac{1}{3}}k^{\frac{5}{3}}\Gamma(k,t). \tag{3.10}$$

Here C_S will turn out to correspond to the universal constant of the scalar spectrum. This form is dimensionally correct and its linearity in Γ respects the dimension [g²]/s of the triple correlation term in (3.6). Similar arguments may be used for the three source terms in (3.6) and their Fourier transforms in (3.8). The dimension of all these terms is [g²]/s. Closure of these terms therefore dictates that they should be linear in Γ. Since S_1 and S_2 involve gradients of G in correlation space, their Fourier transform should contain the wavenumber k. Finally, S_3 contains velocity fluctuation gradients of dimension s^{-1}. It should therefore, like the transfer term, be proportional to $\epsilon^{\frac{1}{3}}$. Dimensional analysis then suggests the following closure assumptions:

$$\left.\begin{aligned} 8\pi k^2\hat{S}_1 &= c_1\,C_S^{-1}\,k\Gamma, \\ 8\pi k^2\hat{S}_2 &= c_2\,C_S^{-1}\,k^2\Gamma, \\ 8\pi k^2\hat{S}_3 &= c_3\,C_S^{-1}\,\epsilon^{\frac{1}{3}}k^{\frac{5}{3}}\Gamma, \end{aligned}\right\} \tag{3.11}$$

where c_1, c_2 and c_3 are empirical constants.

It should be noted that since S_1, S_2 and S_3 in (3.6) are nonlinear, their Fourier transforms should contain contributions not only from k, but from all other wavenumbers as well. Therefore, just as in the closure hypothesis for $W(k,t)$ we have assumed that the interaction between largely separated wavenumbers is negligible compared to the local interaction. The resulting linear differential equation

$$C_S\frac{\partial\Gamma}{\partial t} + \epsilon^{\frac{1}{3}}\left(\tfrac{5}{3}k^{\frac{2}{3}}\Gamma + k^{\frac{5}{3}}\frac{\partial\Gamma}{\partial k}\right) + c_1\,s_L^0\,k\Gamma + c_2\,\mathscr{D}_{\mathscr{L}}\,k^2\Gamma + c_3\,\mathscr{L}\epsilon^{\frac{1}{3}}k^{\frac{5}{3}}\Gamma = 0 \tag{3.12}$$

can be solved exactly by the method of characteristics to yield

$$\Gamma(k,t) = BH(\xi)\,k^{-\frac{5}{3}}\exp\left[-3c_1(L_G\,k)^{\frac{1}{3}}\right]\exp\left[-\tfrac{3}{4}c_2(L_C\,k)^{\frac{4}{3}}\right]\exp\left(-c_3\,\mathscr{L}k\right). \tag{3.13}$$

Here B is a constant and $H(\xi)$ is an unknown function to be determined by initial conditions.

According to the universal equilibrium theory postulated above, Γ should be independent of time in the inertial subrange. This suggests that $H(\xi)$ should approach a constant at $k = k_0(t)$. For simplicity of the subsequent calculations we will consider initial conditions where $H(\xi)$ is a Heaviside step function:

$$H(\xi) = \begin{cases} 1 & \text{for} \quad k \geq k_0 \\ 0 & \text{for} \quad k < k_0 \end{cases}. \tag{3.14}$$

This corresponds to a small-wavenumber cutoff of the scalar spectrum at the integral lengthscale. For the present purpose, such a cutoff is no better and no worse than any

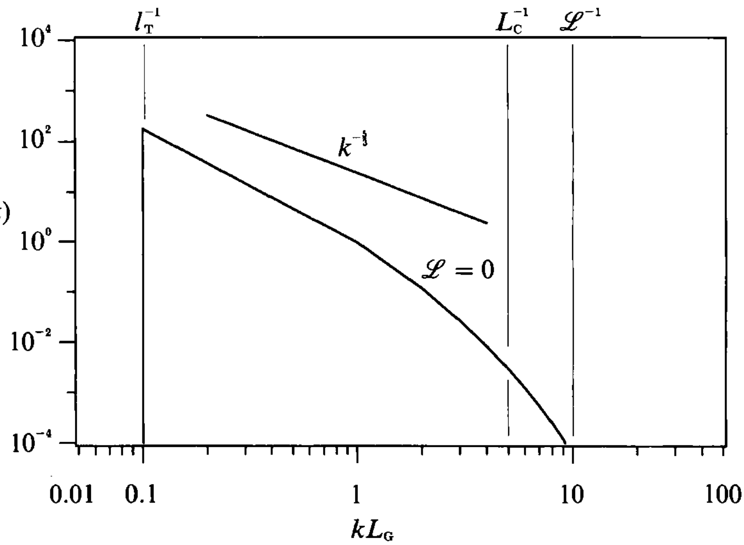

FIGURE 2. The scalar spectrum $\Gamma(k)$ as a function of the non-dimensional wavenumber kL_G, where L_G is the Gibson length, l_T the integral length, L_C the Corrsin length and $\mathscr{L}$ the Markstein length.

other assumption about the large-scale range of the spectrum, which cannot be assumed universal.

The statistics of the turbulent flow field are therefore represented by ϵ and l_T, which for the homogeneous problem considered here, may both depend on time. The resulting scalar spectrum contains an inertial subrange between k_0 and the Gibson wavenumber k_G, and three exponential ranges with different powers of the wavenumber, the first extending from k_G to $k_C = L_C^{-1}$ with an exponential $k^{\frac{1}{3}}$ dependence, the second from k_C to $k_{\mathscr{L}} = \mathscr{L}^{-1}$ with an exponential $k^{\frac{4}{3}}$ dependence and the last one from $k_{\mathscr{L}}$ to infinity with a linear k-dependence within the exponent. The scalar spectrum is schematically plotted in figure 2.

The dominating feature of the present analysis is the cutoff of the inertial subrange at the Gibson scale L_G. This cutoff is due to the term $2s_L^0 S_1$ in (3.2), which represents the two-point correlation between the scalar fluctuation G' and the fluctuation of the source term $s_L^0 \sigma'$ in (2.5). For a one-point correlation this term corresponds to the kinematic restoration introduced above. It has a similar effect in smoothing the scalar fluctuations as the scalar dissipation has for a diffusive scalar. It acts as a sink term in the equation for the scalar variance. This has led to the minus sign in the definition (2.10). For the constant-density analysis in this section we have the relation

$$\omega = 2s_L^0 S_1(r = 0, t). \tag{3.15}$$

The term involving S_2, on the other hand, is active only at a smaller scale, the Corrsin scale L_C. Since it represents the two-point correlation between scalar fluctuations and the curvature term in (1.18), in accounts for the smoothing of the cusps in the flame front. The one-point correlation was called scalar dissipation and can be related to S_2 by

$$\chi_{\mathscr{L}} = 2\mathscr{D}_{\mathscr{L}} S_2(r = 0, t). \tag{3.16}$$

Finally, the term involving S_3, representing a two-point correlation between the scalar fluctuation and the flow divergence term in (1.18), may be related to the scalar-strain co-variance in the constant-density case by

$$\Sigma_{\mathscr{L}} = 2\mathscr{L} S_3(r = 0, t). \tag{3.17}$$

The question arises of whether the resulting exponential decay laws in the scalar spectrum function are physically meaningful. There exists an analogy with the low-Prandtl-number analysis of Batchelor, Howells & Townsend (1959). In the small-Prandtl-number case for a diffusive scalar, the inertial subrange of the energy spectrum extends down to the Kolmolgorov scale $l_K = (\nu^3/\epsilon)^{\frac{1}{4}}$ beyond that of the scalar spectrum, which is cutoff at the Corrsin scale. Since in this case

$$L_C = Pr^{-\frac{3}{4}} l_K, \quad k_C = Pr^{\frac{3}{4}} k_K, \quad k_K = l_K^{-1}, \tag{3.18}$$

with $Pr = \rho \nu c_p / \lambda$, ν being the kinematic viscosity, there is a range $k_C \ll k \ll k_K$ where the scalar is strongly smoothed by diffusion. In the present case, the inertial subrange of the scalar field is cut off at the Gibson length, whereas the inertial subrange of the turbulent kinetic energy continues down to the Kolmogorov scale.

Equation (3.12) may be integrated over k between $k = k_0$ and $k = \infty$. The spectral transfer term then vanishes and by comparison with the constant-density homogeneous turbulence analogue of (2.9)

$$\frac{\partial \overline{G'^2}}{\partial t} = -\omega - \chi_{\mathscr{L}} - \Sigma_{\mathscr{L}} \tag{3.19}$$

one obtains the integrals

$$\overline{G'^2} = \int_{k_0}^{\infty} \Gamma \, \mathrm{d}k, \tag{3.20}$$

$$\omega = c_1 C_s^{-1} s_L^0 \int_{k_0}^{\infty} k \Gamma \, \mathrm{d}k, \tag{3.21}$$

$$\chi_{\mathscr{L}} = c_2 C_s^{-1} \mathscr{D}_{\mathscr{L}} \int_{k_0}^{\infty} k^2 \Gamma \, \mathrm{d}k, \tag{3.22}$$

$$\Sigma_{\mathscr{L}} = c_3 C_s^{-1} \mathscr{L} \int_{k_0}^{\infty} k^{\frac{5}{3}} \Gamma \, \mathrm{d}k. \tag{3.23}$$

When these integrals are written in terms of the variable

$$x = 3c_1 (L_G k)^{\frac{1}{3}} \tag{3.24}$$

and the solution (3.13) with assumption (3.14) is used, one obtains for $\overline{G'^2}$ and ω

$$\overline{G'^2} = 27 c_1^2 L_G^{\frac{2}{3}} B \int_{x_0}^{\infty} x^{-3} \exp\left[-x - \frac{\mathscr{L}}{L_G}\left(\frac{\alpha x^4}{24} + \frac{\beta x^3}{6}\right)\right] \mathrm{d}x, \tag{3.25}$$

$$\omega = C_s^{-1} \epsilon^{\frac{1}{3}} B \int_{x_0}^{\infty} \exp\left[-x - \frac{\mathscr{L}}{L_G}\left(\frac{\alpha x^4}{24} + \frac{\beta x^3}{6}\right)\right] \mathrm{d}x, \tag{3.26}$$

where $$\alpha = 2c_2/(9c_1^4), \quad \beta = 2c_3/(9c_1^3). \tag{3.27}$$

In order to obtain an estimate for the integration constant B, we consider the two limits

$$\mathscr{L} \ll L_G \ll l_T, \tag{3.28}$$

the second condition implying $s_L^0 \ll u'$ and $x_0 \to 0$. Then, with $\mathscr{L}/L_G \to 0$, the integral in (3.26) becomes unity and

$$B = C_s \omega \epsilon^{-\frac{1}{3}}. \tag{3.29}$$

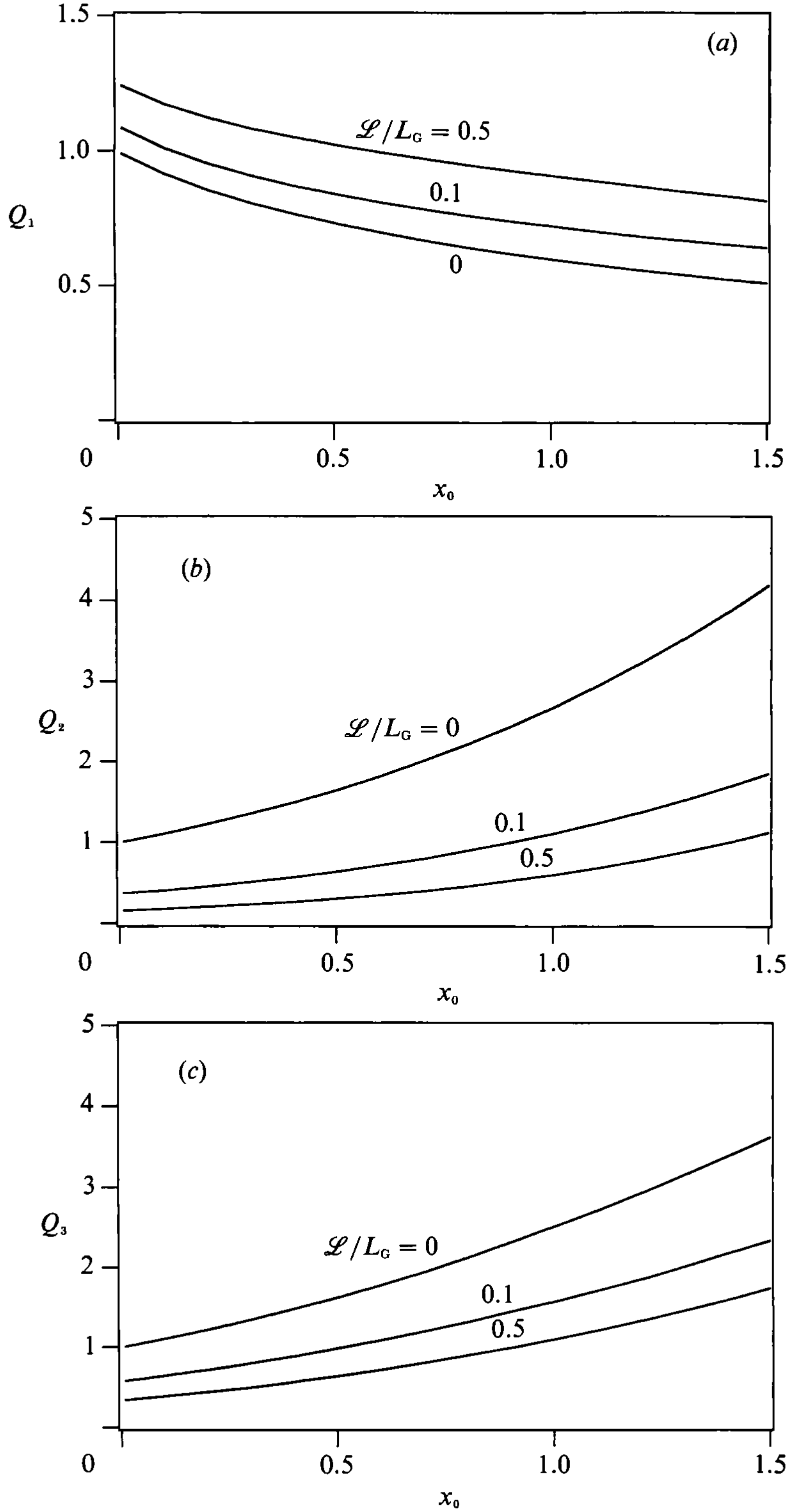

FIGURE 3. The quotient (a) Q_1, (b) Q_2 and (c) Q_3 as functions of x_0 for different values of $\mathscr{L}/L_G$.

In the inertial subrange the scalar spectrum function is then

$$\Gamma(k) = C_s\,\omega\epsilon^{-\frac{1}{3}}k^{-\frac{5}{3}}. \tag{3.30}$$

By comparison with the spectrum function of a diffusive scalar in the inertial subrange

$$\Gamma(k) = C_T\,\chi\epsilon^{-\frac{1}{3}}k^{-\frac{5}{3}} \tag{3.31}$$

it is seen that very similar behaviour is found here, the kinematic restoration ω replacing the scalar dissipation χ. It may therefore be assumed that C_s and C_T have the same numerical values.

Eliminating B between (3.25) and (3.26), one obtains a relation between $\overline{G'^2}$ and ω that may be written as

$$\omega = \frac{2}{3Q_1 C_s}\frac{\epsilon^{\frac{1}{3}}}{l_T^{\frac{2}{3}}}\overline{G'^2}, \tag{3.32}$$

where

$$Q_1\left(x_0, \frac{\mathscr{L}}{L_G}\right) = \frac{I_1}{I_0}, \tag{3.33}$$

$$\left.\begin{aligned} I_0 &= \int_{x_0}^{\infty} \exp\left[-x - \frac{\mathscr{L}}{L_G}\left(\frac{\alpha x^4}{24} + \frac{\beta x^3}{6}\right)\right]\mathrm{d}x, \\ I_1 &= 2x_0^2\int_{x_0}^{\infty} x^{-3}\exp\left[-x - \frac{\mathscr{L}}{L_G}\left(\frac{\alpha x^4}{24} + \frac{\beta x^3}{6}\right)\right]\mathrm{d}x. \end{aligned}\right\} \tag{3.34}$$

The quotient of integrals Q_1 is plotted in figure 3(*a*). It was normalized such that it approaches unity in the limits $x_0 \rightarrow 0$ and $\mathscr{L}/L_G \rightarrow 0$.

If we take $\epsilon = c_d u'^3/l_T$, $c_d = 0.37$ (Bray 1990), the turbulent kinetic energy $\bar{k} = \frac{3}{2}u'^2$ and $C_s = 1.2$, which is the value preferred by Monin & Yaglom (1975) for the universal constant C_T of the scalar spectrum, we obtain with $Q_1 = 1$ for (3.32)

$$\omega = c_\omega \frac{\epsilon}{\bar{k}}\overline{G'^2}, \tag{3.35}$$

where $c_\omega = 1.62$. This relates the kinematic restoration in a similar way to the turbulent timescale $t_T = \bar{k}/\epsilon$ and the scalar fluctuations $\overline{G'^2}$, as usually assumed for the scalar dissipation of a diffusive scalar. Also, using the same relations between ϵ, u' and l_T, x_0 may be expressed as

$$x_0 = 3c_1\left(\frac{L_G}{l_T}\right)^{\frac{1}{3}} = c_0\frac{s_L^0}{u'}, \tag{3.36}$$

where $c_0 = 3c_d^{-\frac{1}{3}}c_1$.

Dividing (3.22) and (3.23) by (3.21) one obtains the following relations:

$$\chi_{\mathscr{L}} = \alpha Q_2\frac{\mathscr{L}}{L_G}\omega, \tag{3.37}$$

where

$$Q_2\left(x_0, \frac{\mathscr{L}}{L_G}\right) = \frac{I_2}{6I_0}, \tag{3.38}$$

$$I_2 = \int_{x_0}^{\infty} x^3\exp\left[-x - \frac{\mathscr{L}}{L_G}\left(\frac{\alpha x^4}{24} + \frac{\beta x^3}{6}\right)\right]\mathrm{d}x, \tag{3.39}$$

and
$$\Sigma_{\mathscr{L}} = \beta Q_3 \frac{\mathscr{L}}{L_G}\omega, \tag{3.40}$$

where
$$Q_3\left(x_0, \frac{\mathscr{L}}{L_G}\right) = \frac{I_3}{2I_0}, \tag{3.41}$$

$$I_3 = \int_{x_0}^{\infty} x^2 \exp\left[-x - \frac{\mathscr{L}}{L_G}\left(\frac{\alpha x^4}{24} + \frac{\beta x^3}{6}\right)\right] \mathrm{d}x. \tag{3.42}$$

The quotients Q_2 and Q_3 approach unity in the limit $x_0 \to 0$, $\mathscr{L}/L_G \to 0$. They are plotted for various values of $\mathscr{L}/L_G$ in figures 3(*b*) and 3(*c*).

4. Modelling of the source terms in the equation for $\bar{G}$

Among the three source terms to be modelled in (2.7) the first one

$$P_1 = s_L^0 \bar{\sigma} \tag{4.1}$$

is the most important. Nothing can be said about this term except that it should be related to ω by dimensional arguments. The situation resembles that of modelling the turbulent viscosity in the equation for the mean velocity by dimensional analysis, once the equations for ϵ and k have been closed. We expect P_1 to be independent of s_L^0 to leading order. This implies that as $s_L^0 \to 0$, $\bar{\sigma}$ should be inversely proportional to s_L^0. Following Kerstein, Ashurst & Williams (1988), $\bar{\sigma}$ represents the flame surface area per unit volume. The physical argument to justify the above assumption goes back as far as to Damköhler (1940) who viewed the turbulent premixed flame as an ensemble of Bunsen flame cones. In fact, when the burning velocity in an idealized Bunsen flame tends to zero, the flame surface area increases as $1/s_L^0$ since the mass flow rate $\dot{m}$ of premixed gas is always burnt, if there is a closed tip of the Bunsen cone. In terms of the flamelet surface-to-volume ratio, the argument was recently substantiated by Bray (1990) who showed that when this ratio is expressed by the inverse of the lengthscale L_y, L_y/l_t is proportional to s_L^0/u'. From dimensional analysis in terms of [g] it follows, since P_1 has the dimension [g]/s and ω the dimension [g^2]/s, that P_1 should be proportional to $\omega^{\frac{1}{2}}$. If s_L^0 is excluded in this dimensional analysis, only the turbulent time $\bar{k}/\epsilon$ is available to provide the correct time dimension of P_1. A dimensionally correct formulation is therefore

$$P_1 = a_1\left(\omega\frac{\epsilon}{\bar{k}}\right)^{\frac{1}{2}}, \tag{4.2}$$

where a_1 is an empirical constant.

It is useful to go one step further and use here the equation for the variance (2.8). It is consistent with standard arguments in turbulence modelling to consider the limit where production equals the sum of the kinematic restoration, scalar dissipation and the scalar-strain term in the variance equation. Such arguments lead to Prandtl's mixing-length theory in turbulent shear flows and to the eddy break-up limit, where the mean reaction rate is proportional to the scalar dissipation rate, in progress-variable descriptions of premixed turbulent combustion. Assuming a gradient flux approximation in the production term

$$-\overline{v'_\alpha G'} = D_T \frac{\partial \bar{G}}{\partial x_\alpha}, \tag{4.3}$$

where the turbulent diffusivity is

$$D_{\mathrm{T}} = c_1 \bar{k}^2/\epsilon \tag{4.4}$$

and c_1 is a constant, this limit yields

$$2D_{\mathrm{T}}|\nabla \bar{G}|^2 = \omega + \chi_{\mathscr{L}} + \Sigma_{\mathscr{L}}. \tag{4.5}$$

Let us at first consider the limit $\mathscr{L} \rightarrow 0$ so that the last two terms in (4.5) can be neglected. Then from (4.4) and (4.5) one obtains

$$\omega = 2c_1 \bar{k}^2 |\nabla \bar{G}|^2/\epsilon. \tag{4.6}$$

Substituting this into (2.8), one obtains

$$\frac{\partial \bar{G}}{\partial t} + \bar{v}_\alpha \frac{\partial \bar{G}}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}(\overline{v'_\alpha G'}) = b_1 u' |\nabla \bar{G}| - \overline{\mathscr{D}_{\mathscr{L}} \kappa |\nabla G|} + \overline{\mathscr{L} \boldsymbol{n} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{n} |\nabla G|}, \tag{4.7}$$

where $b_1 = a_1(3c_1)^{\frac{1}{2}}$ and the curvature and flow divergence terms have been expressed explicitly following (1.14). As noted above, v_α and u' are conditioned velocities in the unburnt gas ahead of the flame. The first term on the right-hand side of (4.7) is exactly analogous to that in (1.18) except that s_{L}^0 is replaced by a constant times the turbulence intensity u'. In a homogeneous turbulent flow field the mean contour of a turbulent premixed flame could then be calculated in a way similar to a laminar flame with the laminar burning velocity replaced by u'. This is consistent with renormalization arguments (Yakhot 1988*a*). Since the equation for G is a Hamilton–Jacobi equation with a parabolic curvature term, its turbulent counterpart, again following renormalization arguments, should have the same character in order to be consistent with the boundary and initial conditions. This suggests that the terms in (4.7), which represent turbulent transport and Markstein diffusion, should be combined and modelled as a curvature term

$$-\frac{\partial}{\partial x_\alpha}(\overline{v'_\alpha G'}) - \overline{\mathscr{D}_{\mathscr{L}} \kappa |\nabla G|} = -D_{\mathrm{T}}^{\kappa} \bar{\kappa} |\nabla \bar{G}|. \tag{4.8}$$

Here $\bar{\kappa}$ is the mean curvature defined as in (1.15) but with $\bar{G}$ instead of G. The turbulent diffusion coefficient D_{T}^{κ} for the curvature term should be of the order of D_{T} defined in (4.4).

The last term in (4.7) is expected to be important in situations where the ratio of the Markstein length to the integral lengthscale is not very small. Since this is a higher-order term in the original equation it is difficult to develop a convincing model for the limit $u'/s_{\mathrm{L}}^0 \rightarrow \infty$ considered here. We will leave the investigation of this term to analysis based on direct numerical simulation.

5. Model calculations for steady normal and oblique flames in a constant-velocity flow with homogeneous non-decaying turbulence

For plane and oblique flames such as a V-shaped flame the curvature term disappears. For simplicity, we will also neglect the last term, representing stretch due to flow divergence, in (4.7). It may be interesting to derive the solution for such steady-state flames in a constant-velocity flow by assuming all turbulent quantities to be constant. We consider a two-dimensional (x, y)-coordinate system with the flow velocity v in the y direction. For a plane one-dimensional flame normal to the flow v is equal to the burning turbulent velocity $s_{\mathrm{T,n}}$. The solution of (4.7) is then $\bar{G} = cy$ where c is an arbitrary constant, leading to

$$s_{\mathrm{T,n}} = b_1 u'. \tag{5.1}$$

This is the classical result due to Damköhler (1940). Deviations from that limit occur at finite values of $\mathscr{L}/l_{\mathrm{T}}$. They are due to flame stretch by flow divergence resulting from the last term in (4.7). The turbulent flame brush thickness $l_{\mathrm{F,T}}$ may be calculated from the variance equation. It may be defined as

$$l_{\mathrm{F,T}} = \frac{(\overline{G'^2})^{\frac{1}{2}}}{|\nabla \bar{G}|}. \tag{5.2}$$

For the plane flame normal to the flow the variance equation (2.8) becomes

$$v\frac{\partial \overline{G'^2}}{\partial y} = \frac{\partial}{\partial y}\left(D_{\mathrm{T},2}\frac{\partial \overline{G'^2}}{\partial y}\right) + 2D_{\mathrm{T}}|\nabla \bar{G}|^2 - c_\omega \frac{\epsilon}{\bar{k}}\overline{G'^2}\left[1+(\alpha Q_2+\beta Q_3)\frac{\mathscr{L}}{l_{\mathrm{T}}}\right], \tag{5.3}$$

where (4.3), (3.35), (3.37) and (3.40) has been used. Also, a gradient flux approximation has been introduced for the turbulent transport terms in (2.8):

$$-\frac{\partial}{\partial x_\alpha}(\overline{v'_\alpha G'^2}) = \frac{\partial}{\partial x_\alpha}\left(D_{\mathrm{T},2}\frac{\partial \overline{G'^2}}{\partial x_\alpha}\right). \tag{5.4}$$

Since the plane turbulent flame should be translationally invariant, the solution $\overline{G'^2}$ = const leads to a balance of production and kinematic restoration:

$$\overline{G'^2} = \frac{2c^2 D_{\mathrm{T}}\bar{k}}{c_\omega \ \epsilon}\left[1+(\alpha Q_2+\beta Q_3)\frac{\mathscr{L}}{l_{\mathrm{T}}}\right]^{-1}, \tag{5.5}$$

where the solution $\bar{G} = cy$ has been introduced. The arbitrary constant c cancels in the definition of the flame brush thickness. Since the grouping $D_{\mathrm{T}}\bar{k}/\epsilon$ is proportional to l_{T}^2 the flame brush thickness $l_{\mathrm{F,T,n}}$, of a steady turbulent flame normal to the flow is proportional to and of the order of the integral lengthscale l_{T}. Relating the turbulent diffusivity D_{T} to the eddy viscosity $\nu_{\mathrm{T}} = c_{\mathrm{d}}\bar{k}^2/\epsilon$ as $D_{\mathrm{T}} = \nu_{\mathrm{T}}/Sc$, where Sc is a turbulent Schmidt number, one obtains

$$l_{\mathrm{F,T,n}} = l_{\mathrm{T}}\left(\frac{27c_D}{4c_\omega c_{\mathrm{d}}^2 Sc}\right)^{\frac{1}{2}}\left[1+(\alpha Q_2+\beta Q_3)\frac{\mathscr{L}}{l_{\mathrm{T}}}\right]^{-\frac{1}{2}}. \tag{5.6}$$

The term in square brackets describes the influence of the Markstein length on the flame thickness. A Lewis number larger than one leading to a large Markstein number would increase this term and therefore reduce the flame brush thickness as compared to a case where the Lewis number is smaller than one and the Markstein number small. This is consistent with the picture that a large Markstein length would smooth the local flame front and annihilate cusps, thereby reducing the flame brush thickness in a turbulent flow. Such an effect was observed experimentally in hydrogen flames with different equivalence ratios by Wu *et al.* (1991).

The V-shaped flame also to be considered is assumed to be attached to the origin $x = 0$, $y = y_0$, where x is the coordinate normal to the flow. The solution of (4.7) is now

$$\bar{G} = c\left(-\frac{(v^2 - s_{\mathrm{T,n}}^2)^{\frac{1}{2}}}{s_{\mathrm{T,n}}}|x| + y - y_0\right), \tag{5.7}$$

where c is again an arbitrary constant. The flame contour $\bar{G} = G_0 = 0$ is obtained as

$$y = y_0 + \frac{(v^2 - s_{\mathrm{T,n}}^2)^{\frac{1}{2}}}{s_{\mathrm{T,n}}}|x|. \tag{5.8}$$

Since y_0 is arbitrary, the flame brush thickness must be independent of y. With $|\nabla \bar{G}|^2$ being constant for this flame one may use (2.8) to write an equation for the flame brush thickness as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(D_{\mathrm{T},2}\frac{\mathrm{d}l_{\mathrm{F,t}}^2}{\mathrm{d}x}\right)+2D_{\mathrm{T}}-c_{\omega}\frac{\epsilon}{\bar{k}}l_{\mathrm{F,T}}^2\left[1+\alpha Q_2+\beta Q_3)\frac{\mathscr{L}}{l_{\mathrm{T}}}\right]=0. \tag{5.9}$$

Assuming that the diffusivities $D_{\mathrm{T},2}$ and D_{T} are related to each other by

$$D_{\mathrm{T}} = d_2^2 D_{\mathrm{T},2}, \tag{5.10}$$

where d_2 is constant, one obtains with the boundary conditions $l_{\mathrm{F,T}} = 0$ at $x = 0$, $\mathrm{d}l_{\mathrm{F,t}}/\mathrm{d}x = 0$ for $x \rightarrow \infty$ the solution

$$l_{\mathrm{F,t}} = l_{\mathrm{F,t,n}}[1-\exp(-d_2 x/l_{\mathrm{F,T,n}})]^{\frac{1}{2}}. \tag{5.11}$$

For small values of x this shows a square root increase of the flame brush thickness for V-shaped flames, as it is approximately observed experimentally. It approaches the constant thickness of the normal flame when x times d_2 equals $l_{\mathrm{F,T,n}}$. If the flame angle is small, this would happen far downstream from the point of attachment where the assumption of constant properties of turbulence may no longer be valid. The flame brush thickness will then increase at the rate of the turbulence lengthscale in decaying turbulence. It should be noted that the solution (5.11), although formally correct, ceases to be valid for values of x smaller than the integral lengthscale. Close to the point of attachment only eddies of the order of x can interact with the flame front and displace it. This phenomenon is not accounted for by a description of turbulence based on integral length- and timescales only.

Since the last term in (4.7) has not been modelled, no attempt will be made here to compare this theory with experimental data for $s_{\mathrm{T}}/s_{\mathrm{L}}$. Extensive compilations of such data have been provided by Abdel-Gayed & Bradley (1981), Abdel-Gayed, Al-Kishali & Bradley (1984), Abdel-Gayed *et al.* (1985) and Abdel-Gayed, Bradley & Lawes (1987). While the general tendencies of these data do not seem to contradict the present results, the correspondence between parameters used in the experiments and those that appear in the present theory is not easily established. There are still large uncertainties about the adequate calculation of the Markstein length as well as the definition of the laminar flame thickness, and there are large errors in the experimental data concerning the integral lengthscale and even the turbulence intensity.

Furthermore, in many experimental situations a steady turbulent flame is not established and transient effects may be important. The assumption of a homogeneous turbulent flow field whose turbulence properties can be represented by u' and l_{T} only is certainly also not always valid. These shortcomings, however, could be overcome if the appropriate unsteady and (or) non-homogeneous solution of (4.7) and (2.8) were compared with experimental data.

6. Discussion and conclusions

The present analysis tries to combine a systematic analysis of the properties of the scalar field $G(\boldsymbol{x}, t)$ in isotropic turbulence with intuitive arguments from previous work on premixed turbulent combustion. A basic quantity in this context is the Gibson scale L_{G}, which reappears naturally from dimensional analysis in the modelled equation for the scalar spectrum $\Gamma(k, t)$. The cutoff of the inertial subrange at the Gibson scale due to kinematic restoration is fairly weak, but it is enhanced by the flame curvature effect at the Corrsin scale L_C and by flame stretch effects at the Markstein length $\mathscr{L}$.

The details of the analysis involve a number of empirical parameters, which result from the modelling of the nonlinear terms in (1.18). It would therefore be desirable to measure instantaneous flame contours and flame brush thicknesses and the velocity field simultaneously and to evaluate these constants. An evaluation of the source terms in the equation for $\bar{G}$ would be even more desirable, but probably even more difficult to perform.

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